

The smallest 3-uniform bi-hypergraphs which are one-realization of a given set

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Abstract

For any set S of positive integers, a mixed hypergraph \mathcal{H} is a one-realization of S if its feasible set is S and each entry of its chromatic spectrum is either 0 or 1. In this paper, we determine the minimum size of 3-uniform bi-hypergraphs which are one-realizations of a given set S . As a result, we partially solve an open problem proposed by Bujtás and Tuza in 2008.

Key words: mixed hypergraph; feasible set; chromatic spectrum; one-realization

1 Introduction

A *mixed hypergraph* on a finite set X is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where \mathcal{C} and \mathcal{D} are families of subsets of X . The members of \mathcal{C} and \mathcal{D} are called *\mathcal{C} -edges* and *\mathcal{D} -edges*, respectively. A set $B \in \mathcal{C} \cap \mathcal{D}$ is called a *bi-edge*. A *bi-hypergraph* is a mixed hypergraph with $\mathcal{C} = \mathcal{D}$, denoted by $\mathcal{H} = (X, \mathcal{B})$, where $\mathcal{B} = \mathcal{C} = \mathcal{D}$. If $\mathcal{C}' = \{C \in \mathcal{C} | C \subseteq X'\}$ and $\mathcal{D}' = \{D \in \mathcal{D} | D \subseteq X'\}$, then the hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ is called a *derived sub-hypergraph* of \mathcal{H} on X' , denoted by $\mathcal{H}[X']$.

The distinction between \mathcal{C} -edges and \mathcal{D} -edges becomes substantial when colorings are considered. A *proper k -coloring* of \mathcal{H} is a partition of X into k *color classes* such that each \mathcal{C} -edge has two vertices with a *Common* color and each \mathcal{D} -edge has two vertices with *Distinct* colors. A *strict k -coloring* is a proper k -coloring with k nonempty color classes, and a mixed hypergraph is *k -colorable* if it has a strict k -coloring. For more information, see [4, 5, 6]. The set of all the values k such that \mathcal{H} has a strict k -coloring is called the *feasible set* of \mathcal{H} , denoted by $\Phi(\mathcal{H})$. For each k , let r_k denote the number of *partitions* of the vertex set. The vector $R(\mathcal{H}) = (r_1, r_2, \dots, r_{\bar{\chi}})$ is called the *chromatic spectrum* of \mathcal{H} , where $\bar{\chi}$ is the largest possible number of colors in a strict coloring of \mathcal{H} . If S is a finite set of positive integers, we say that a mixed hypergraph \mathcal{H} is a *realization* of S if $\Phi(\mathcal{H}) = S$. A mixed hypergraph \mathcal{H} is a *one-realization* of S if it is a realization of S and all the entries of the chromatic spectrum of \mathcal{H} are either 0 or 1. This concept was firstly introduced by Král [3].

It is readily seen that if $1 \in \Phi(\mathcal{H})$, then \mathcal{H} cannot have any \mathcal{D} -edges. Let S be a finite set of positive integers with $\min(S) \geq 2$. Jiang et al. [2] proved that the

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minimum number of vertices of realizations of S is $2 \max(S) - \min(S)$ if $|S| = 2$ and $\max(S) - 1 \notin S$. Král [3] proved that there exists a one-realization of S with at most $|S| + 2 \max S - \min S$ vertices. In [8], we improved Král's result and proved that, for $\min(S) \geq 3$, the smallest size of the one-realizations of S is $2 \max(S) - \min(S)$ if $\max(S) - 1 \notin S$ or $2 \max(S) - \min(S) - 1$ if $\max(S) - 1 \in S$. Recently, Bujtás and Tuza [1] gave a necessary and sufficient condition for S being the feasible set of an r -uniform mixed hypergraph, and they raised the following open problem:

Problem. Determine the minimum number of vertices in r -uniform bi-hypergraphs with a given feasible set.

In [7], we constructed a family of 3-uniform bi-hypergraphs with a given feasible set, and obtained an upper bound on the minimum number of vertices of the one-realizations of a given set. In this paper, we focus on this problem and obtain the following result:

Theorem 1.1 *For integers $s \geq 2$ and $n_1 > n_2 > \dots > n_s \geq 2$, the smallest size of 3-uniform bi-hypergraphs which are one-realizations of $\{n_1, n_2, \dots, n_s\}$ is $2n_1 - \lfloor \frac{n_2+1}{n_1} \rfloor$.*

2 Proof of Theorem 1.1

In this section we always assume that $S = \{n_1, n_2, \dots, n_s\}$ is a set of integers with $s \geq 2$ and $n_1 > n_2 > \dots > n_s \geq 2$. In order to prove our main result, we first give a lower bound on the size of the 3-uniform bi-hypergraphs which are one-realizations of S , then construct two families of 3-uniform bi-hypergraphs which meet the bound.

Lemma 2.1 *Let $\delta_3(S)$ denote the minimum size of 3-uniform bi-hypergraphs $\mathcal{H} = (X, \mathcal{B})$ which are one-realizations of S . Then $\delta_3(S) \geq 2n_1 - \lfloor \frac{n_2+1}{n_1} \rfloor$.*

Proof. We divide our proof into the following two cases.

Case 1 $n_1 > n_2 + 1$.

That is to say, $n_1 - 1 \notin S$. Suppose $|X| = 2n_1 - 1$. For any strict n_1 -coloring $c = \{C_1, C_2, \dots, C_{n_1}\}$ of \mathcal{H} , if there exist two color classes, say C_1 and C_2 , such that $|C_1| = |C_2| = 1$, then $c' = \{C_1 \cup C_2, C_3, \dots, C_{n_1}\}$ is a strict $(n_1 - 1)$ -coloring of \mathcal{H} , a contradiction. Since $|X| = 2n_1 - 1$, there exists one color class, say $C_1 \in c$, such that $|C_1| = 1$, and $|C_i| = 2$ for any $i = 2, 3, \dots, n_1$. Suppose $C_1 = \{x\}$ and $C_i = \{x_i, y_i\}, i = 2, 3, \dots, n_1$. Then $c_1 = \{\{x, x_2, x_3, \dots, x_{n_1}\}, \{y_2, y_3, \dots, y_{n_1}\}\}$ and $c_2 = \{\{x_2, x_3, \dots, x_{n_1}\}, \{x, y_2, y_3, \dots, y_{n_1}\}\}$ are two distinct strict 2-colorings of \mathcal{H} , a contradiction. If $|X| \leq 2n_1 - 2$, then we can get a strict $(n_1 - 1)$ -coloring of \mathcal{H} from a strict n_1 -coloring of \mathcal{H} , also a contradiction.

Case 2 $n_1 = n_2 + 1$.

Suppose $|X| = 2n_1 - 2$. For any strict n_1 -coloring $c = \{C_1, C_2, \dots, C_{n_1}\}$ of \mathcal{H} , if there exist three color classes, say C_1, C_2 and C_3 , such that $|C_1| = |C_2| = |C_3| = 1$, then $c' = \{C_1 \cup C_2, C_3, \dots, C_{n_1}\}$ and $c'' = \{C_1, C_2 \cup C_3, C_4, \dots, C_{n_1}\}$ are two distinct strict n_2 -colorings of \mathcal{H} , a contradiction. Referring that $|X| = 2n_1 - 2$, there exist two color classes each of which has one vertex, and each of the other color classes has two vertices. Similar to Case 1, \mathcal{H} has two distinct strict 2-colorings, a contradiction. If $|X| \leq 2n_1 - 3$, we can also have a contradiction.

Hence, the desired result follows. \square

In the rest, we shall construct two families of 3-uniform bi-hypergraphs which meet the bound in Lemma 2.1.

We first construct the desired 3-uniform bi-hypergraphs for the case of $n_1 > n_2 + 1$. For any positive integer n , let $[n]$ denote the set $\{1, 2, \dots, n\}$.

Construction I. For $i \in [s] \setminus \{1\}$, write

$$\begin{aligned} X_s^s &= \bigcup_{j=1}^{n_s} \{(\underbrace{j, j, \dots, j}_s, 0), (\underbrace{j, j, \dots, j}_s, 1)\}, \\ X_{i-1}^s &= \bigcup_{k=0}^{n_{i-1}-n_i-1} \{(\underbrace{n_i+k, \dots, n_i+k}_{i-1}, \underbrace{1, \dots, 1}_{s-i+1}, 0), (\underbrace{n_i+k, \dots, n_i+k}_{i-1}, n_i, \dots, n_s, 1)\}, \\ X_{n_1, \dots, n_s} &= \{(n_1, n_2, \dots, n_s, 1)\} \cup \bigcup_{t=1}^s X_t^s, \\ \mathcal{B}_{n_1, \dots, n_s} &= \{ \{\alpha_1, \alpha_2, \alpha_3\} \mid \alpha_l \in X_{n_1, \dots, n_s}, l \in [3], |\{\alpha_{1(j)}, \alpha_{2(j)}, \alpha_{3(j)}\}| = 2, j \in [s+1] \} \\ &\quad \cup \{ (1, \dots, 1, 1, 0), (n_s, \dots, n_s, 1, 0), (n_s, \dots, n_s, n_s, 0) \} \}, \end{aligned}$$

where $\alpha_{l(j)}$ is the j -th entry of the vertex α_l . Then $\mathcal{H}_{n_1, \dots, n_s} = (X_{n_1, \dots, n_s}, \mathcal{B}_{n_1, \dots, n_s})$ is a 3-uniform bi-hypergraph.

Note that for any $i \in [s]$, $c_i^s = \{X_{i1}^s, X_{i2}^s, \dots, X_{in_i}^s\}$ is a strict n_i -coloring of $\mathcal{H}_{n_1, \dots, n_s}$, where X_{ij}^s consists of vertices $(x_1, \dots, x_{i-1}, j, x_{i+1}, \dots, x_s, x)$.

In the following we shall prove that c_1^s, \dots, c_s^s are all the strict colorings of $\mathcal{H}_{n_1, \dots, n_s}$ by induction on s .

Lemma 2.2 \mathcal{H}_{n_1, n_2} is a one-realization of $\{n_1, n_2\}$.

Proof. Let $c = \{C_1, C_2, \dots, C_m\}$ be a strict coloring of \mathcal{H}_{n_1, n_2} . We focuss on the colors of $(1, 1, 0)$, $(1, 1, 1)$, and have the following two possible cases.

Case 1 $(1, 1, 0)$ and $(1, 1, 1)$ fall into a common color class.

Suppose $(1, 1, 0), (1, 1, 1) \in C_1$. From the bi-edges

$$\{(2, 2, 0), (1, 1, 0), (1, 1, 1)\}, \{(2, 2, 1), (1, 1, 1), (1, 1, 0)\}, \{(2, 2, 0), (2, 2, 1), (1, 1, 1)\},$$

we have $(2, 2, 0), (2, 2, 1) \notin C_1$ and $(2, 2, 0)$ and $(2, 2, 1)$ fall into a common color class, say C_2 . Similarly, we have $(j, j, 0), (j, j, 1) \in C_j$ for any $j \in [n_2]$. The bi-edge $\{(n_2, 1, 0), (1, 1, 0), (n_2, n_2, 0)\}$ implies that $(n_2, 1, 0) \in C_1 \cup C_{n_2}$.

Case 1.1 $(n_2, 1, 0) \in C_1$.

The bi-edges $\{(n_2+k, n_2, 1), (1, 1, 1), (1, 1, 0)\}, \{(n_2+k, n_2, 1), (n_2, n_2, 1), (n_2, 1, 0)\}$ imply that $(n_2+k, n_2, 1) \in C_{n_2}$ for any $k \in [n_1 - n_2]$. Since $\{(n_2+k, 1, 0), (n_2+k, n_2, 1), (n_2, n_2, 1)\}, \{(n_2+k, 1, 0), (n_2+k, n_2, 1), (1, 1, 0)\}$ are bi-edges, $(n_2+k, 1, 0) \in C_1$ for any $k \in [n_1 - n_2 - 1]$, and so $c = c_2^2$.

Case 1.2 $(n_2, 1, 0) \in C_{n_2}$.

For any $j \in [n_2 - 1]$ and $k \in [n_1 - n_2 - 1]$, the bi-edge $\{(n_2+k, n_2, 1), (j, j, 1), (j, j, 0)\}$ implies that $(n_2+k, n_2, 1) \notin C_j$; from the bi-edge $\{(n_2+k, n_2, 1), (n_2, 1, 0), (n_2, n_2, 0)\}$,

we have $(n_2+k, n_2, 1) \notin C_{n_2}$. Suppose $(n_2+1, n_2, 1) \in C_{n_2+1}$. Since $\{(n_2+1, 1, 0), (n_2+1, n_2, 1), (n_2, 1, 0)\}, \{(n_2+1, 1, 0), (n_2, 1, 0), (n_2, n_2, 1)\}$ are bi-edges, $(n_2+1, 1, 0) \in C_{n_2+1}$. Similarly, for any $k \in [n_1-n_2-1]$, $(n_2+k, 1, 0), (n_2+k, n_2, 1) \in C_{n_2+k}$. For any $j \in [n_2-1]$, the bi-edge $\{(n_1, n_2, 1), (j, j, 0), (j, j, 1)\}$ implies that $(n_1, n_2, 1) \notin C_j$; and for any $k \in [n_1-n_2-1] \cup \{0\}$, from the bi-edge $\{(n_1, n_2, 1), (n_2+k, 1, 0), (n_2+k, n_2, 1)\}$, we have $(n_1, n_2, 1) \notin C_{n_2+k}$. Then $(n_1, n_2, 1) \in C_{n_1}$, and $c = c_1^2$.

Case 2 $(1, 1, 0)$ and $(1, 1, 1)$ fall into distinct color classes.

Suppose $(1, 1, 0) \in C_1, (1, 1, 1) \in C_2$. From the bi-edge $\{(n_2, n_2, 0), (1, 1, 0), (1, 1, 1)\}$, we have $(n_2, n_2, 0) \in C_1 \cup C_2$. Suppose $(n_2, n_2, 0) \in C_1$. The bi-edges

$$\begin{aligned} & \{(n_2, n_2, 1), (1, 1, 1), (1, 1, 0)\}, \{(n_2, n_2, 1), (n_2, n_2, 0), (1, 1, 0)\}, \\ & \{(n_2, 1, 0), (n_2, n_2, 0), (1, 1, 1)\}, \{(n_2, 1, 0), (n_2, n_2, 0), (1, 1, 0)\} \end{aligned}$$

imply that $(n_2, n_2, 1) \in C_2$ and $(n_2, 1, 0) \in C_2$. Therefore, the three vertices of the bi-edge $\{(n_2, 1, 0), (n_2, n_2, 1), (1, 1, 1)\}$ fall into a common color class, a contradiction. Suppose $(n_2, n_2, 0) \in C_2$. Similarly, we also have a contradiction. It follows that Case 2 does not appear. \square

Lemma 2.3 $\mathcal{H}_{n_1, n_2, n_3}$ is a one-realization of $\{n_1, n_2, n_3\}$.

Proof. Let $X'_{n_1, n_2, n_3} = X_3^3 \cup X_2^3 \cup \{(n_2, n_2, n_3, 1)\}$, $\mathcal{H}'_{n_1, n_2, n_3} = \mathcal{H}_{n_1, n_2, n_3}[X'_{n_1, n_2, n_3}]$. Then

$$\phi: X'_{n_1, n_2, n_3} \longrightarrow X_{n_2, n_3}, \quad (x_2, x_2, x_3, x) \longmapsto (x_2, x_3, x)$$

is an isomorphism from $\mathcal{H}'_{n_1, n_2, n_3}$ to \mathcal{H}_{n_2, n_3} .

For any strict coloring $c = \{C_1, C_2, \dots, C_m\}$ of $\mathcal{H}_{n_1, n_2, n_3}$, since the restriction of any strict coloring of $\mathcal{H}_{n_1, n_2, n_3}$ on X'_{n_1, n_2, n_3} corresponds to a strict coloring of \mathcal{H}_{n_2, n_3} , by Lemma 2.2 we get the following two possible cases.

Case 1 $(j, j, j, 0), (j, j, j, 1) \in C_j$ for each $j \in [n_3]$, $(n_3+k, n_3+k, 1, 0) \in C_1$ for any $k \in [n_2-n_3-1] \cup \{0\}$, and $(n_3+k, n_3+k, n_3, 1) \in C_{n_3}$ for any $k \in [n_2-n_3]$.

In this case, we shall prove that $c = c_3^3$. It suffices to discuss the colors of the vertices in $X_1^3 \cup \{(n_1, n_2, n_3, 1)\}$. The bi-edges $\{(n_2+k, 1, 1, 0), (n_3, n_3, 1, 0), (n_3, n_3, n_3, 1)\}, \{(n_2+k, 1, 1, 0), (n_3, n_3, n_3, 0), (n_3, n_3, n_3, 1)\}$ imply that $(n_2+k, 1, 1, 0) \in C_1$ for any $k \in [n_1-n_2-1] \cup \{0\}$, and from the bi-edges $\{(n_2+k, n_2, n_3, 1), (1, 1, 1, 1), (1, 1, 1, 0)\}, \{(n_2+k, n_2, n_3, 1), (n_3, n_3, n_3, 1), (n_3, n_3, 1, 0)\}$, we have $(n_2+k, n_2, n_3, 1) \in C_{n_3}$ for any $k \in [n_1-n_2]$. Therefore, $c = c_3^3$.

Case 2 $(j, j, j, 0), (j, j, j, 1) \in C_j$ for each $j \in [n_3]$, $(n_3+k, n_3+k, 1, 0), (n_3+k, n_3+k, n_3, 1) \in C_{n_3+k}$ for any $k \in [n_2-n_3-1] \cup \{0\}$ and $(n_2, n_2, n_3, 1) \in C_{n_2}$.

Then we shall prove that $c = c_1^3$ or $c = c_2^3$. Since $\{(n_2, n_2, n_3, 1), (n_2, 1, 1, 0), (1, 1, 1, 0)\}$ is a bi-edge, we have $(n_2, 1, 1, 0) \in C_1 \cup C_{n_2}$.

Case 2.1 $(n_2, 1, 1, 0) \in C_1$.

For any $k \in [n_1-n_2]$, since $\{(n_2+k, n_2, n_3, 1), (n_2, n_2, n_3, 1), (n_2, 1, 1, 0)\}, \{(n_2+k, n_2, n_3, 1), (1, 1, 1, 0), (1, 1, 1, 1)\}$ are bi-edges, $(n_2+k, n_2, n_3, 1) \in C_{n_2}$. For any $k \in [n_1-n_2-1]$, from the bi-edges $\{(n_2+k, 1, 1, 0), (n_2, 1, 1, 0), (n_2, n_2, n_3, 1)\}, \{(n_2+k, 1, 1, 0), (n_2+k, n_2, n_3, 1), (n_2, n_2, n_3, 1)\}$, we have $(n_2+k, 1, 1, 0) \in C_1$. Therefore, $c = c_1^3$.

Case 2.2 $(n_2, 1, 1, 0) \in C_{n_2}$.

The bi-edge $\{(n_2 + 1, n_2, n_3, 1), (j, j, j, 0), (j, j, j, 1)\}$ implies that $(n_2 + 1, n_2, n_3, 1) \notin C_j$ for any $j \in [n_3 - 1]$; from the bi-edge $\{(n_2 + 1, n_2, n_3, 1), (n_3 + k, n_3 + k, 1, 0), (n_3 + k, n_3 + k, n_3, 1)\}$, we have $(n_2 + 1, n_2, n_3, 1) \notin C_{n_3+k}$ for any $k \in [n_2 - n_3 - 1] \cup \{0\}$. Since $\{(n_2 + 1, n_2, n_3, 1), (n_2, 1, 1, 0), (n_2, n_2, n_3, 1)\}$ is a bi-edge, $(n_2 + 1, n_2, n_3, 1) \notin C_{n_2}$. Suppose $(n_2 + 1, n_2, n_3, 1) \in C_{n_2+1}$. Then the bi-edges $\{(n_2 + 1, 1, 1, 0), (n_2 + 1, n_2, n_3, 1), (n_2, 1, 1, 0)\}$, $\{(n_2 + 1, 1, 1, 0), (n_2, 1, 1, 0), (n_2, n_2, n_3, 1)\}$ imply that $(n_2 + 1, 1, 1, 0) \in C_{n_2+1}$. Similarly, for any $k \in [n_1 - n_2 - 1]$, $(n_2 + k, 1, 1, 0), (n_2 + k, n_2, n_3, 1) \in C_{n_2+k}$, and $(n_1, n_2, n_3, 1) \in C_{n_1}$. Therefore, $c = c_1^3$. \square

Theorem 2.4 $\mathcal{H}_{n_1, \dots, n_s}$ is a one-realization of $\{n_1, n_2, \dots, n_s\}$.

Proof. By Lemma 2.2 and Lemma 2.3, the conclusion is true for $s = 2$ and $s = 3$. Assume that it is also true for the case of $s - 1$.

Let $X'_{n_1, \dots, n_s} = \bigcup_{i=2}^s X_i^s \cup \{(n_2, n_2, n_3, \dots, n_s, 1)\}$, $\mathcal{H}'_{n_1, \dots, n_s} = \mathcal{H}_{n_1, \dots, n_s}[X'_{n_1, \dots, n_s}]$. Then

$$\psi : X'_{n_1, \dots, n_s} \longrightarrow X_{n_2, n_3, \dots, n_s}, \quad (x_2, x_2, x_3, \dots, x_s, x) \longmapsto (x_2, x_3, \dots, x_s, x)$$

is an isomorphism from $\mathcal{H}'_{n_1, \dots, n_s}$ to $\mathcal{H}_{n_2, n_3, \dots, n_s}$. By induction, all the strict colorings of $\mathcal{H}'_{n_1, \dots, n_s}$ are as follows:

$$c'_i = \{X'_{i1}, X'_{i2}, \dots, X'_{in_i}\}, \quad i \in [s] \setminus \{1\},$$

where $X'_{ij} = X'_{n_1, \dots, n_s} \cap X_{ij}^s$, $j \in [n_i]$.

For any strict coloring $c = \{C_1, C_2, \dots, C_m\}$ of $\mathcal{H}_{n_1, \dots, n_s}$, since the restriction on X'_{n_1, \dots, n_s} of any strict coloring of $\mathcal{H}_{n_1, \dots, n_s}$ corresponds to a strict coloring of $\mathcal{H}_{n_2, n_3, \dots, n_s}$, we focus on the restriction of c on X'_{n_1, \dots, n_s} and get the following two possible cases:

Case 1 $c|_{X'_{n_1, \dots, n_s}} = c'_2$.

That is to say $(j, j, x_3, \dots, x_s, x) \in C_j$ for any $j \in [n_2]$ and $(j, j, x_3, \dots, x_s, x) \in X'_{n_1, \dots, n_s}$. In this case, we shall prove that $c = c_1^s$ or $c = c_2^s$. It suffices to discuss the colors of the vertices in $X_1 \cup \{(n_1, n_2, \dots, n_s, 1)\}$. We obtain $(n_2, 1, \dots, 1, 0) \in C_1 \cup C_{n_2}$ from the bi-edge $\{(n_2, 1, \dots, 1, 0), (n_2, n_2, n_3, \dots, n_s, 1), (1, \dots, 1, 0)\}$.

Case 1.1 $(n_2, 1, \dots, 1, 0) \in C_1$.

For any $k \in [n_1 - n_2 - 1]$, from the bi-edges

$$\begin{aligned} & \{(n_2 + k, n_2, \dots, n_s, 1), (n_2, n_2, n_3, \dots, n_s, 1), (n_2, 1, \dots, 1, 0)\}, \\ & \{(n_2 + k, n_2, \dots, n_s, 1), (1, 1, \dots, 1, 1), (1, 1, \dots, 1, 0)\}, \\ & \{(n_2 + k, 1, \dots, 1, 0), (n_2 + k, n_2, \dots, n_s, 1), (n_2, n_2, n_3, \dots, n_s, 1)\}, \\ & \{(n_2 + k, 1, \dots, 1, 0), (n_2 + k, n_2, \dots, n_s, 1), (n_2, 1, \dots, 1, 0)\}, \end{aligned}$$

we have $(n_2 + k, n_2, \dots, n_s, 1) \in C_{n_2}$ and $(n_2 + k, 1, \dots, 1, 0) \in C_1$. The bi-edges

$$\begin{aligned} & \{(n_1, n_2, \dots, n_s, 1), (n_2, n_2, n_3, \dots, n_s, 1), (n_2, 1, \dots, 1, 0)\}, \\ & \{(n_1, n_2, \dots, n_s, 1), (1, 1, \dots, 1, 1), (1, 1, \dots, 1, 0)\} \end{aligned}$$

imply that $(n_1, n_2, \dots, n_s, 1) \in C_{n_2}$. Therefore, $c = c_2^s$.

Case 1.2 $(n_2, 1, \dots, 1, 0) \in C_{n_2}$.

For any $j \in [n_s - 1]$, the bi-edge $\{(n_2 + 1, n_2, \dots, n_s, 1), (j, \dots, j, 0), (j, \dots, j, 1)\}$ implies that $(n_2 + 1, n_2, \dots, n_s, 1) \notin C_j$. For any $p \in [s] \setminus \{1, 2\}$ and $k \in [n_{p-1} - n_p - 1] \cup \{0\}$, from the bi-edge $\{(n_2 + 1, n_2, \dots, n_s, 1), (n_p + k, \dots, n_p + k, n_p, \dots, n_s, 1), (n_p + k, \dots, n_p + k, 1, \dots, 1, 0)\}$, we have $(n_2 + 1, n_2, \dots, n_s, 1) \notin C_{n_p+k}$; and the bi-edge $\{(n_2 + 1, n_2, \dots, n_s, 1), (n_2, n_2, n_3, \dots, n_s, 1), (n_2, 1, \dots, 1, 0)\}$ implies $(n_2 + 1, n_2, \dots, n_s, 1) \notin C_{n_2}$. Suppose $(n_2 + 1, n_2, \dots, n_s, 1) \in C_{n_2+1}$. From the bi-edges

$$\begin{aligned} &\{(n_2 + 1, 1, \dots, 1, 0), (n_2, 1, \dots, 1, 0), (n_2, n_2, n_3, \dots, n_s, 1)\}, \\ &\{(n_2 + 1, 1, \dots, 1, 0), (n_2 + 1, n_2, n_3, \dots, n_s, 1), (n_2, n_2, n_3, \dots, n_s, 1)\}, \end{aligned}$$

we have $(n_2 + 1, 1, \dots, 1, 0) \in C_{n_2+1}$. Similarly, for any $k \in [n_1 - n_2 - 1]$, $(n_2 + k, 1, \dots, 1, 0), (n_2 + k, n_2, n_3, \dots, n_s, 1) \in C_{n_2+k}$, furthermore, $(n_1, n_2, n_3, \dots, n_s, 1) \in C_{n_1}$. Therefore, $c = c_1^s$.

Case 2 $c|_{X'_{n_1, \dots, n_s}} = c'_p$ for some $p \in [s] \setminus \{1, 2\}$.

That is to say, $(x_2, x_2, x_3, \dots, x_{p-1}, j, x_{p+1}, \dots, x_s, x) \in C_j$ for any $j \in [n_p]$ and $(x_2, x_2, x_3, \dots, x_{p-1}, j, x_{p+1}, \dots, x_s, x) \in X'_{n_1, \dots, n_s}$. For any $k \in [n_1 - n_2]$, we have $(n_2 + k, n_2, n_3, \dots, n_s, 1) \in C_{n_p}$ from the bi-edges

$$\begin{aligned} &\{(n_2 + k, n_2, \dots, n_p, \dots, n_s, 1), (1, 1, \dots, 1, 1), (1, 1, \dots, 1, 0)\}, \\ &\{(n_2 + k, n_2, \dots, n_p, \dots, n_s, 1), (n_p, \dots, n_p, n_{p+1}, \dots, n_s, 1), (\underbrace{n_p, \dots, n_p}_{p-1}, 1, \dots, 1, 0)\}. \end{aligned}$$

Then, for any $k \in [n_1 - n_2 - 1]$, from the bi-edges

$$\begin{aligned} &\{(n_2 + k, 1, \dots, 1, 0), (n_2 + k, n_2, n_3, \dots, n_s, 1), (1, 1, \dots, 1, 0)\}, \\ &\{(n_2 + k, 1, \dots, 1, 0), (n_2 + k, n_2, n_3, \dots, n_s, 1), (n_2, n_2, n_3, \dots, n_s, 1)\}, \end{aligned}$$

we have $(n_2 + k, 1, \dots, 1, 0) \in C_1$. Since

$$\begin{aligned} &\{(n_2, 1, \dots, 1, 0), (n_2, n_2, n_3, \dots, n_s, 1), (1, 1, \dots, 1, 0)\}, \\ &\{(n_2, 1, \dots, 1, 0), (n_2, n_2, n_3, \dots, n_s, 1), (n_2 + 1, n_2, n_3, \dots, n_s, 1)\} \end{aligned}$$

are bi-edges, $(n_2, 1, \dots, 1, 0) \in C_1$. Therefore, $c = c_p^s$. \square

For the case of $n_2 = n_1 - 1$, we have the following construction.

Construction II. Let $X''_{n_1, \dots, n_s} = X_{n_1, \dots, n_s} \setminus \{(n_2, 1, \dots, 1, 0)\}$ and $\mathcal{H}''_{n_1, \dots, n_s} = \mathcal{H}_{n_1, \dots, n_s}[X'']$. Then, for any $i \in [s]$,

$$c''_i = \{X''_{i1}, X''_{i2}, \dots, X''_{in_i}\}$$

is a strict n_i -coloring of $\mathcal{H}''_{n_1, \dots, n_s}$, where $X''_{ij} = X''_{n_1, \dots, n_s} \cap X^s_{ij}, j \in [n_i]$.

Theorem 2.5 $\mathcal{H}''_{n_1, \dots, n_s}$ is a one-realization of $\{n_1, n_2, \dots, n_s\}$.

Proof. For any strict coloring $c = \{C_1, C_2, \dots, C_m\}$ of \mathcal{H}'' , referring to the proof of Theorem 2.4, there are the following two possible cases:

Case 1 $c|_{X'_{n_1, \dots, n_s}} = c'_2$.

That is to say, $(j, j, x_3, \dots, x_s, x) \in C_j$ for any $j \in [n_2]$ and $(j, j, x_3, \dots, x_s, x) \in X'$. Similar to the Case 1 of Theorem 2.4, we have $(n_1, n_2, \dots, n_s, 1) \in C_{n_2} \cup C_{n_1}$. Therefore, $c = c_2^s$ if $(n_1, n_2, \dots, n_s, 1) \in C_{n_2}$ and $c = c_1^s$ if $(n_1, n_2, \dots, n_s, 1) \in C_{n_1}$.

Case 2 $c|_{X'_{n_1, \dots, n_s}} = c'_p$ for some $p \in [s] \setminus \{1, 2\}$.

The bi-edges

$$\begin{aligned} & \{(n_1, \dots, n_p, n_{p+1}, \dots, n_s, 1), (1, 1, \dots, 1, 1), (1, 1, \dots, 1, 0)\} \\ & \{(n_1, \dots, n_p, n_{p+1}, \dots, n_s, 1), (n_p, \dots, n_p, n_{p+1}, \dots, n_s, 1), \underbrace{(n_p, \dots, n_p, 1, \dots, 1, 0)}_{p-1}\} \end{aligned}$$

imply that $(n_1, n_2, n_3, \dots, n_s, 1) \in C_{n_p}$. Therefore, $c = c_p^s$. \square

Observe $|X_{n_1, \dots, n_s}| = 2n_1$ and $|X'_{n_1, \dots, n_s}| = 2n_1 - 1$. Combining Lemma 2.1, Lemma 2.2, Theorems 2.4 and Theorem 2.5, the proof of Theorem 1.1 is completed.

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